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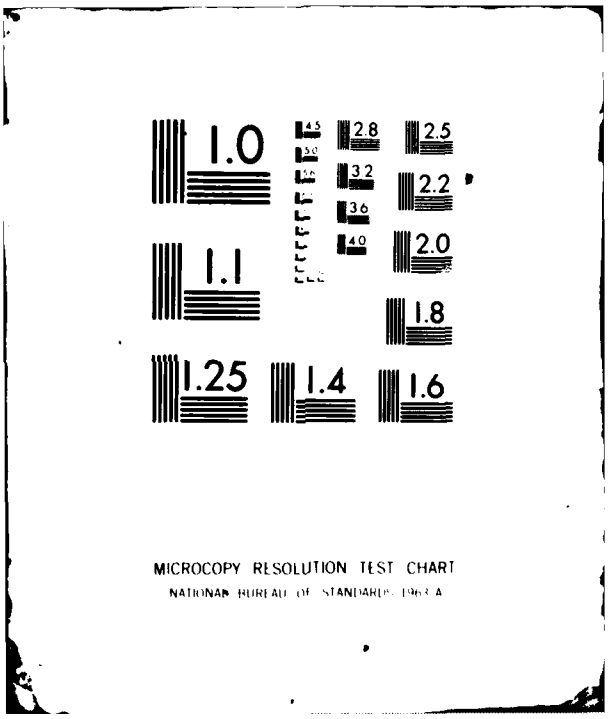
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Recursive least squares spectral estimation algorithms have attracted much attention recently because of their excellent convergence behavior and fast parameter tracking capability. They are recursive in the sense that as a new element of the time series is observed, the parameters of a spectral estimation model are algorithmically updated. Recently, a recursive algorithm for efficiently obtaining an autoregressive (AR) spectral estimate has been introduced by Morf and Lee [2]. In this paper a more insightful development of their technique is presented. A modofication of the data, namely prewindowing, is applied to achieve a significant,

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computational improvement. The development is predicated on utilization of a projection operator.

RECURSIVE AR SPECTRAL ESTIMATION

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ABSTRACT

Recursive least squares spectral estimation algorithms have attracted much attention recently because of their excellent convergence behavior and fast parameter tracking capability. They are recursive in the sense that as a new element of the time series is observed, the parameters of a spectral estimation model are algorithmically updated. Recently, a recursive algorithm for efficiently obtaining an autoregressive (AR) spectral estimate has been introduced by Morf and Lee [2]. In this paper a more insightful development of their technique is presented. A modification of the data, namely prewindowing, is applied to achieve a significant computational improvement. The development is predicated on utilization of a projection operator.

I. INTRODUCTION

In many relevant signal processing applications, one seeks to characterize the spectral density of a time series based upon a finite set of time series observations. Without loss of generality, this sample observation set is taken to be the contiguous set of N measurements as given by

$$x(1),\ x(2),\ \cdots,\ x(N) \tag{1.1}$$

One of the most widely used spectral estimation models is obtained by postulating the following autoregressive (AR) structure

$$x(n) + a_1 x(n-1) + \cdots + a_m x(n-m) = \epsilon(n) \tag{1.2}$$

in which $\epsilon(n)$ is a white noise time series with zero mean and variance σ_ϵ^2. Our object will be that of modeling an underlying time series $\{x(n)\}$ with the AR model structure (1.2) in which the a_k coefficients are estimated from the given finite set of observations (1.1). This is readily achieved by applying the well known one-step prediction.

An m-th order one-step predictor, by definition, estimates the value of a random time series using a linear combination of the most recent m samples. Namely, the sample $x(n)$ is estimated by means of the relationship

$$\hat{x}(n) = -\sum_{k=1}^{m} a_k x(n-k) \tag{1.3}$$

The difference between this predicted value and the observed value x(n) over the observation interval is called the prediction error and is specified by

$$e(n) = x(n) - \hat{x}(n) \qquad m < n \leq N \tag{1.4}$$

or

$$e(n) = x(n) + \sum_{k=1}^{m} a_k x(n-k) \qquad m < n \leq N \tag{1.5}$$

Writing these error expressions in matrix form yields

$$\underline{e} = \underline{x} + X\underline{a}$$

where $\underline{a}$, $\underline{e}$, and $\underline{x}$ are m×1, (N-m)×1, and (N-m)×1 column vectors, respectively, given by

$$\underline{a} = [a_1, \cdots, a_m]^T \tag{1.7a}$$

$$\underline{e} = [e(m+1), e(m+2), \cdots, e(N)]^T \tag{1.7b}$$

$$\underline{x} = [x(m+1), x(m+2), \cdots, x(N)]^T \tag{1.7c}$$

and X is an (N-m)×m matrix specified by

$$X = \begin{bmatrix} x(m), & x(m+1), & \cdots, & x(N-1) \\ x(m-1), & x(m), & \cdots, & x(N-2) \\ \vdots & \vdots & & \vdots \\ x(1), & x(2), & \cdots, & x(N-m) \end{bmatrix}^T \tag{1.7d}$$

where the superscript T denotes the transpose operation.

The a_k coefficients are to be now selected so as to cause each of the predictor error terms e(n) to be close to zero. This selection process will give rise to the so-called optimal one-step predictor. To achieve the required objective of setting the e(n) to be near zero, one typically appeals to the least squares method which minimizes a squared error criterion of the form

$$f(\underline{a}) = \underline{e}^T w \underline{e} \tag{1.8}$$

where w is an $(N-m)\times(N-m)$ nonnegative definite square matrix. The minimization of this quadratic functional with respect to the column vector $\underline{a}$ is straightforwardly carried out and results in

$$X^T w X \underline{a}^\circ = X^T w \underline{x} \tag{1.9}$$

It can be shown that the resulting power spectral density estimate of the time series $\{x(n)\}$ is then given by [1],

$$S_X(w) = \frac{\sigma_\varepsilon^2}{|\ 1+a_1^\circ e^{-jw}+a_2^\circ e^{-2jw}+ \cdots +a_p^\circ e^{-pjw}|^2} \tag{1.10}$$

where the a_k° coefficients are obtained upon solving relationship (1.9). Generally the solution of relationship (1.9) requires on the order of m^3(i.e.$o(m^3)$) number of multiplications and additions if that relationship is directly used. This computational requirement can be excessive in many real time applications. It has been recently shown by Morf and Lee [2] that this computational requirement can be reduced to $o(m)$ by slightly reformulating the matrix X and column vector $\underline{x}$. In many interesting cases, fortunately, the solution to this modified system of equations will be close to that of the desired solution as represented by expression (1.9). In this paper, the method which is identical to the LMS algorithm of Morf and Lee [2] is presented with more emphasis on insightful development.

This general modification methodology shall herein be referred to as data modification. Applying the specific data modification method referred to as prewindowing, the matrix X is reformulated as the $N\times m$ matrix given by

$$X = \begin{bmatrix} 0 & x(1) & x(2) & \cdots & x(m) & \cdots & x(N-1) \\ 0 & 0 & x(1) & \cdots & x(m-1) & \cdots & x(N-2) \\ \vdots & \vdots & 0 & & \vdots & & \vdots \\ \vdots & \vdots & \vdots\ \ddots & & \vdots & & \vdots \\ 0 & 0 & 0 \cdots 0 & & x(1) & \cdots & x(N-m) \end{bmatrix}^T \tag{1.11}$$

while the $N\times 1$ column vector $\underline{x}$ is specified by

$$\underline{x} = [x(1), x(2), \cdots, x(N)]^T \tag{1.12}$$

If these new entrants are substituted into relationship (1.9), an efficient solution procedure for $\underline{a}^\circ$ is possible. The structure of this reformulated matrix X and the column vector $\underline{x}$ enables us to obtain a recursive least square spectral estimation algorithm which has an

excellent convergence behavior and a fast parameter tracking capability relative to the former structure. The development of this algorithm is predicated on the utilization of projection operator theory [3]. In the sections which follow the necessary projection operator theory to be used in the algorithm is described.

II. VECTOR SPACE FORMULATION

In this section, the given spectral estimation problem will be cast into a convenient vector space setting. It will be assumed that the following observations of the time series $\{x(n)\}$ as specified by

$$x(1), x(2), \cdots, x(N) \tag{2.1}$$

are given. This in turn will give rise to the associated column data vector

$$\underline{x}_N = [x(1), x(2), \cdots, x(N)]^T \tag{2.2}$$

The vector $\underline{x}_N$ lies in the product space

$$H_N = R \times R \times \cdots \times R = R^N \tag{2.3}$$

This vector space can be made into an inner product space by defining the following inner product between any two elements $\underline{x}_N, \underline{y}_N \in H_N$

$$< \underline{x}_N, \underline{y}_N > = \underline{x}_N^T \underline{y}_N = \sum_{n=1}^{N} x(n)y(n) \tag{2.4}$$

The corresponding induced norm of $\underline{x}_N$ is then given by

$$\| x_N \| = [< \underline{x}_N, \underline{x}_N >]^{\frac{1}{2}} = \left[\sum_{n=1}^{N} x^2(n) \right]^{\frac{1}{2}} \tag{2.5}$$

We next define the shift matrix S which is represented by the $N \times N$ matrix

$$S = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & O \\ & \ddots & \ddots & \\ & & \ddots & \ddots \\ O & & & 1\ 0 \end{bmatrix} \tag{2.6}$$

Applying the shift matrix m times to the column vector $\underline{x}_N$ is seen to yield

$$S^m \underline{x}_N = [\underbrace{0, \cdots, 0}_{m \text{ zeroes}}, x(1), \cdots, x(N-m-1), x(N-m)]^T \tag{2.7}$$

We next construct the subspace $M\underline{x}_{N[i,m]}$ which is spanned by the set of vectors $S^i\underline{x}_N, \cdots, S^m\underline{x}_N$. This subspace will be suggestively denoted by

$$M\underline{x}_{N[i,m]} = \{S^i\underline{x}_N, S^{i+1}\underline{x}_N, \cdots, S^m\underline{x}_N\} \tag{2.8}$$

where the first integer index i may take on any value in the set $\{0,1,\cdots,m\}$. Next, we let $P\underline{x}_{N[i,m]}$ designate the projection operator on the subspace $M\underline{x}_{N[i,m]}$. This projection operator can be shown to have the form

$$P\underline{x}_{N[i,m]} \triangleq X\underline{x}_{N[i,m]} \left[X\underline{x}^T_{N[i,m]} \cdot X\underline{x}_{N[i,m]} \right]^{-1} X\underline{x}_{N[i,m]} \tag{2.9}$$

where $X\underline{x}_{N[i,m]}$ is the $N \times (m-i+1)$ matrix composed of the following ordered set of column vectors

$$X\underline{x}_{N[i,m]} = [S^i\underline{x}_N, S^{i+1}\underline{x}_N, \cdots, S^m\underline{x}_N] \tag{2.10}$$

Similarly, the projection operator on the orthogonal complement of subspace $M\underline{x}_{N[i,m]}$ is denoted by

$$P^{\perp}_{\underline{x}_N}[i,m] = I - P\underline{x}_{N[i,m]} \tag{2.11}$$

where I is the N×N identity matrix. It then follows that

$$P\underline{x}_{N[i,m]}\, \underline{y}_N = \underline{y}_N \qquad \text{if } \underline{y}_N \in M\underline{x}_{N[i,m]} \tag{2.12}$$

$$P^{\perp}_{\underline{x}_{N[i,m]}}\, \underline{y}_N \perp S^k\underline{x}_N \quad i \le k \le m, \text{ if } \underline{y}_N \in H_N \tag{2.13}$$

Expression (2.12) and (2.13) specify those properties of the projection operators which will be utilized when developing a recursive least square algorithm in the next section.

III. LINEAR PREDICTION AND PROJECTION OPERATOR

In this section, we will define three methods of linear prediction, namely, forward prediction, backward prediction, and delayed backward prediction. These projection operators will play a central role in the algorithmic solution procedure to be developed.

(a) Forward Prediction

The m-th order forward prediction method is referred to as that specific procedure for estimating the column vector $\underline{x}_N$ by means of a linear combination of the set of m shifted vectors $\{S^1\underline{x}_N, S^2\underline{x}_N, \cdots, S^m\underline{x}_N\}$. It then follows that the m-th order forward prediction estimate of $\underline{x}_N$ is of the form

$$\hat{\underline{x}}_{N[1,m]} = -\sum_{k=1}^{m} a_k S^k \underline{x}_N \tag{3.1}$$

while the associated forward error vector is specified by

$$\underline{\varepsilon}_{N,m} = \underline{x}_N - \hat{\underline{x}}_{N[1,m]} \tag{3.2a}$$

$$= \underline{x}_N + \sum_{k=1}^{m} a_k S^k \underline{x}_N \tag{3.2b}$$

Upon examination of the structure of the shifted vector $S^k\underline{x}_N (k = 1, \cdots, m)$, expression (3.2b) leads to the aforementioned prewindowing formula where X and $\underline{x}$ are given by (1.11) and (1.12), respectively.

The problem at hand is to then find the scalar constants $a_1, a_2, \cdots, a_m$ which minimize the squared forward prediction error

$$f(\underline{a}) = \| \underline{x}_N - \hat{\underline{x}}_{N[1,m]} \|^2 \tag{3.3}$$

According to the projection theorem [4], $f(\underline{a})$ is minimized when the error vector is orthogonal to each of the one-dimensional subspaces spanned by $S^i\underline{x}_N (i = 1, \cdots, m)$. Thus, we have the orthogonality relationship expressed by

$$(\underline{x}_N - \hat{\underline{x}}_{N[1,m]}) \perp S^i\underline{x}_N \qquad \text{for } i = 1, 2, \cdots, m \tag{3.4}$$

which takes the inner product format

$$< \underline{x}_N - \hat{\underline{x}}_{N[1,m]}, S^i\underline{x}_N > = 0 \quad \text{for } i = 1, 2, \cdots, m \tag{3.5}$$

Substitution of expression (3.1) into (3.5) yields the set of linear algebraic equations

$$\sum_{k=1}^{m} < S^k\underline{x}_N, S^i\underline{x}_N > a_k = - < \underline{x}_N, S^i\underline{x}_N > \tag{3.6}$$

$$\text{for } i = 1, 2, \cdots, m$$

for the optimum set of a_k prediction coefficients. These equations are called the normal equations and can be put into the matrix form

$$\left[\underline{X}_{N[1,m]}^{T}\ \underline{X}_{N[1,m]}\right]\underline{a} = -\underline{X}_{N[1,m]}^{T}\ \underline{x}_N \tag{3.7}$$

where

$$\underline{X}_{N[1,m]} = [S^1\underline{x}_N,\ S^2\underline{x}_N,\ \cdots,\ S^m\underline{x}_N]$$

$$\underline{a} = [a_1,\ a_2,\ \cdots,\ a_m]^T$$

Solving equation (3.7) for $\underline{a}$ and substituting this solution into expression (3.1) then yields the optimum prediction vector

$$\hat{\underline{x}}_{N[1,m]} = \underline{X}_{N[1,m]}\left[\underline{X}_{N[1,m]}^{T}\ \underline{X}_{N[1,m]}\right]^{-1}\underline{X}_{N[1,m]}^{T}\underline{x}_N \tag{3.8}$$

Upon examination of the projection operator (2. 9) and this expression, $\hat{\underline{x}}_{N[1,m]}$ is seen to be compactly specified by

$$\hat{\underline{x}}_{N[1,m]} = P_{\underline{X}_{N[1,m]}}\ \underline{x}_N \tag{3.9}$$

Thus, we see that $\hat{\underline{x}}_{N[1,m]}$ is obtained by projecting $\underline{x}_N$ onto the subspace $M_{\underline{X}_{N[1,m]}}$ and the m-th order forward prediction error vector is obtained by projecting $\underline{x}_N$ onto the orthogonal complement of $M_{\underline{X}_{N[1,m]}}$ in the H_N, that is

$$\underline{\epsilon}_{N,m} = P^{\perp}_{\underline{X}_{N[1,m]}}\ \underline{x}_N \tag{3.10}$$

The corresponding minimum mean squared error is then defined to be

$$f^{\epsilon}_{N,m} = \underline{\epsilon}^{T}_{N,m}\ \underline{\epsilon}_{N,m} = \underline{\epsilon}^{T}_{N,m}\ \underline{x}_N$$

(b) Backward Prediction

The m-th order backward prediction method is that procedure of estimating the m-th shifted column vector $S^m\underline{x}_N$ by a linear combination of the set of shifted vectors $\{S^0\underline{x}_N,\ S^1\underline{x}_N,\ \cdots,\ S^{m-1}\underline{x}_N\}$. This backward estimate is then of the form

$$\hat{\underline{x}}_{N[o,m-1]} = -\sum_{k=o}^{m-1} b_k S^k \underline{x}_N \tag{3.12}$$

and the backward error vector is defined by

$$\underline{b}_{N,m} = S^m \underline{x}_N - \hat{\underline{x}}_{N[o,m-1]} \tag{3.13}$$

In the same manner as with forward prediction, by applying the projection theorem it can be shown that the backward estimate is given by

$$\hat{\underline{x}}_{N[o,m-1]} = P_{\underline{x}_{N[o,m-1]}} S^m \underline{x}_N \tag{3.14}$$

The backward prediction error vector is then found to be

$$\underline{b}_{N,m} = P^{\perp}_{\underline{x}_{N[o,m-1]}} S^m \underline{x}_N \tag{3.15}$$

and the corresponding minimum mean squared error is obtained by

$$f^b_{N,m} = \underline{b}^T_{N,m} \underline{b}_{N,m} = \underline{b}^T_{N,m} S^m \underline{x}_N \tag{3.16}$$

(c) Delayed Backward Prediction

The m-th order delayed backward prediction method is similarly defined to be that procedure of estimating the column vector $S^{m+1}\underline{x}_N$ by a linear combination of the set of vectors $\{S^1\underline{x}_N, S^2\underline{x}_N, \cdots, S^m\underline{x}_N\}$. It can be shown that the delayed backward estimate is given by

$$\hat{\underline{x}}_{N[1,m]} = P_{\underline{x}_{N[1,m]}} S^{m+1} \underline{x}_N \tag{3.17}$$

and the delayed backward error is obtained by

$$\underline{d}_{N,m} = P^{\perp}_{\underline{x}_{N[1,m]}} S^{m+1} \underline{x}_N \tag{3.18}$$

The corresponding minimum mean squared error is measured by

$$f^d_{N,m} = \underline{d}^T_{N,m} \underline{d}_{N,m} = \underline{d}^T_{N,m} S^{m+1} \underline{x}_N \tag{3.19}$$

A little thought will convince oneself that the projection operation $P\underline{x}_{N[1,m]}$ can be expressed as

$$P\underline{x}_{N[1,m]} = X\underline{x}_{N[1,m]}\left[X\underline{x}^{T}_{N[1,m]}\ X\underline{x}_{N[1,m]}\right]^{-1} X\underline{x}^{T}_{N[1,m]}$$

$$= \begin{bmatrix} \underline{o} \\ X\underline{x}_{N-1[o,m-1]} \end{bmatrix}\left[X\underline{x}^{T}_{N-1[o,m-1]}\ X\underline{x}_{N-1[o,m-1]}\right]^{-1}\begin{bmatrix} \underline{o} \\ X\underline{x}_{N-1[o,m-1]} \end{bmatrix}^{T} \tag{3.20}$$

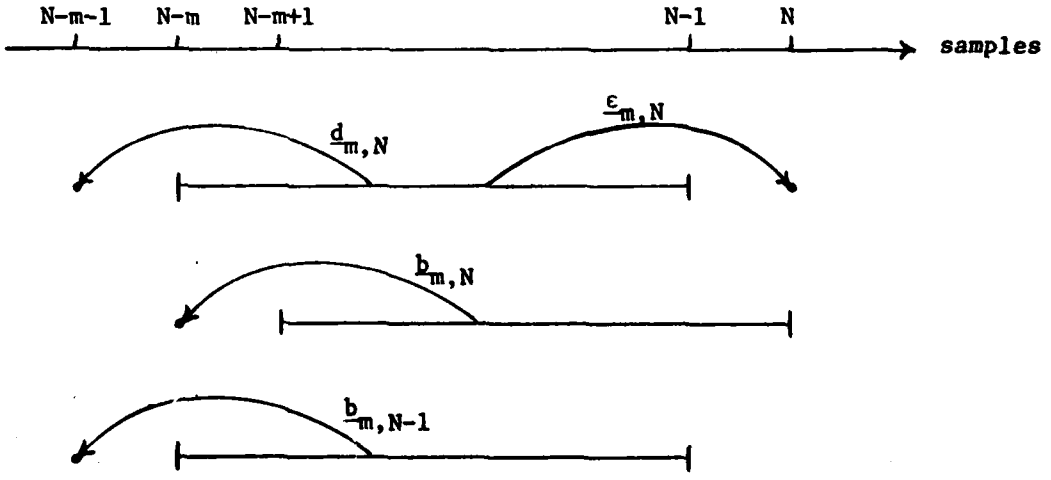
The relationship between the backward prediction error and the delayed backward prediction error is then readily found to be

$$\underline{d}_{N,m} = [o\,;\ \underline{b}_{N-1,m}]^{T} \tag{3.21}$$

It then follows that N^{th} delayed prediction error is equal to the $(N-1)^{st}$ backward prediction error

$$\underline{d}_{N,m}(N) = \underline{b}_{N-1,m}(N-1) \tag{3.22}$$

The relationship of forward, backward, and delayed backward is suggestively depicted in [Fig. 1].



[Fig. 1] Forward, backward and delayed predictions.

IV. DECOMPOSITION OF SUBSPACES

The development of a computational efficient algorithm is dependent on the decomposition of subspaces. Subspaces may be decomposed by appealing to the well known projection theorem [4]. The formulae obtained in this section will be used for the development of order update recursions in section V .

Since the forward prediction error $\underline{\varepsilon}_{N,m}$ lies in the subspace $M\underline{x}_{N[o,m]}$ but is orthogonal to $M\underline{x}_{N[1,m]}$, we can express $M\underline{x}_{N[o,m]}$ as the direct sum of $M\underline{x}_{N[1,m]}$ and $\{\underline{\varepsilon}_{N,m}\}$, that is

$$M\underline{x}_{N,m[o,m]} = M\underline{x}_{N[1,m]} \oplus \{\underline{\varepsilon}_{N,m}\} \tag{4.1}$$

where $\{\underline{\varepsilon}_{N,m}\}$ denotes the subspace spanned by the forward prediction error vector $\underline{\varepsilon}_{N,m}$. The projection operator on the subspace $\{\underline{\varepsilon}_{N,m}\}$ is defined by

$$P\underline{\varepsilon}_{N,m} = \underline{\varepsilon}_{N,m} \left(\underline{\varepsilon}^{T}_{N,m}\, \underline{\varepsilon}_{N,m}\right)^{-1} \underline{\varepsilon}^{T}_{N,m} \tag{4.2}$$

Relationship (4.1) can be readily shown to yield the following decomposition of the projection operator

$$P\underline{x}^{\perp}_{N[o,m]} = (I - P\underline{\varepsilon}_{N,m}) P\underline{x}^{\perp}_{N[1,m]} \tag{4.3}$$

Similarly, since the delayed backward prediction error $\underline{d}_{N,m}$ lies in the subspace $M\underline{x}_{N[1,m+1]}$ but is orthogonal to $M\underline{x}_{N[1,m]}$, we obtain

$$M\underline{x}_{N[1,m+1} = M\underline{x}_{N[1,m]} \oplus \{\underline{d}_{N,m}\} \tag{4.4}$$

where $\{\underline{d}_{N,m}\}$ denotes the subspace spanned by the backward prediction error vector. The projection operator on the subspace $\{\underline{d}_{N,m}\}$ is defined by

$$P\underline{d}_{N,m} = \underline{d}_{N,m} \left(\underline{d}^{T}_{N,m}\, \underline{d}_{N,m}\right)^{-1} \underline{d}^{T}_{N,m} \tag{4.5}$$

Relationship (4.4) is found to yield the following decomposition of the projection operator

$$P\underline{x}^{\perp}_{N[1,m+1]} = (I - P\underline{d}_{N,m}) P\underline{x}^{\perp}_{N[1,m]} \tag{4.6}$$

V. ORDER UPDATE RECURSIONS

In this section, we describe the order update recursive formulae which recursively compute the optimum $m+1^{st}$ order prediction error from the optimum m^{th} order prediction error. Expressions (4.3) and (4.6) play a central role in obtaining these order update recursions.

Let us first derive the order update recursion for the forward prediction error vector. Applying the projection operator (4.6) to the column vector $\underline{x}_N$ yields

$$\underline{\varepsilon}_{N,m+1} = (I - P\underline{d}_{N,m})\,\underline{\varepsilon}_{N,m} \tag{5.1}$$

Substituting expression (4.5) into this relationship then yields

$$\underline{\varepsilon}_{N,m+1} = \underline{\varepsilon}_{N,m} - \underline{d}_{N,m}(\underline{d}_{N,m}^T\,\underline{d}_{N,m})^{-1}\,\underline{d}_{N,m}^T\,\underline{\varepsilon}_{N,m} \tag{5.2}$$

Recalling expression (3.22), the order update recursion for the N^{th} forward prediction error is found to be

$$\underline{\varepsilon}_{N,m+1}(N) = \underline{\varepsilon}_{N,m}(N) - \Delta_{N,m+1}(f_{N-1,m})^{-1}\underline{b}_{N-1,m}(N-1) \tag{5.3}$$

where the partial-correlation coefficients are specified by

$$\Delta_{N,m+1} = \underline{d}_{m,N}^T\,\underline{-m,N} = \underline{x}_N^T\,P\underline{x}^{\perp}_{N[1,m]}\,S^{m+1}\underline{x}_N \tag{5.4}$$

Expression (5.1) leads to

$$\underline{\varepsilon}_{N,m+1}^T\,\underline{\varepsilon}_{N,m+1} = \underline{\varepsilon}_{N,m}^T\,(I - P\underline{d}_{N,m})\underline{\varepsilon}_{N,m} \tag{5.5a}$$

The recursion for the forward minimum mean square error is similarly found to be

$$f^{\varepsilon}_{N,m+1} = f^{\varepsilon}_{N,m} - \Delta_{N,m+1}(f^{b}_{N-1,m})^{-1}\Delta_{N,m+1} \tag{5.5b}$$

Expressions (5.3) and (5.5b) constitute the order update recursion formulae for the forward prediction.

Next, we will find the order update recursion for the backward prediction error vector. Applying the projection operator (4.3) to the

column vector $S^{m+1}\underline{x}_N$ is found to yield

$$\underline{b}_{N,m+1} = (I - P\underline{\varepsilon}_{N,m})\,\underline{d}_{N,m} \tag{5.6}$$

Substituting expression (4.2) into this relationship results in

$$\underline{b}_{N,m+1} = \underline{d}_{N,m} - \underline{\varepsilon}^{T}_{N,m}\,(\underline{\varepsilon}^{T}_{N,m}\,\underline{\varepsilon}_{N,m})^{-1}\,\underline{\varepsilon}^{T}_{N,m}\,\underline{d}_{N,m} \tag{5.7}$$

The order update recursion for the N^{th} backward prediction error is then specified by

$$\underline{b}_{N,m+1}(N) = \underline{b}_{N-1,m}(N-1) - \Delta_{N,m+1}(f^{\varepsilon}_{N,m})^{-1}\underline{\varepsilon}_{N,m}(N) \tag{5.8}$$

Expression (5.6) leads to

$$\underline{b}^{T}_{N,m+1}\,\underline{b}_{N,m+1} = \underline{d}^{T}_{N,m}(I - P\underline{\varepsilon}_{N,m})\,\underline{d}_{N,m} \tag{5.9}$$

The recursion for $f^{b}_{N,m}$ is next found to be

$$f^{b}_{N,m+1} = f^{b}_{N-1,m} - \Delta_{N,m+1}(f^{\varepsilon}_{N,m})^{-1}\Delta_{N,m+1} \tag{5.10}$$

The order update recursion formulae for the backward prediction are represented by relationships (5.8) and (5.10)

VI. TIME UPDATE RECURSIONS

As a new element of the time series is observed, the partial reflection coefficients, forward least square errors, and the backward least square errors can be computed recursively by using the knowledge of these parameters from the last time instance. This being the case, these parameters are said to be "time updated" for each new data point. These update recursions are obtained by utilizing a method referred to as projection operator decomposition.

For the spectral estimation problem considered here, we decompose the projection operator $P\underline{x}_{N[i,m]}$ into one that projects on all past observations and another that generates the correction due to a new observation x(N). First, we define the component projection matrix P_N by

$$P_N = e_N e_N^{T} \tag{6.1}$$

where e_N is the N×1 unit basis vector expressed by

$$e_N = [o, \cdots, o, 1]^T \tag{6.2}$$

Let us define the column vectors

$$\underline{x}_{P_N} = P_N \underline{x}_N = [o, \cdots, o, x(N)]^T \tag{6.3}$$

$$\underline{x}_{P_N^{\perp}} = P_N^{\perp} \underline{x}_N = [x(1), \cdots, x(N-1), o]^T \tag{6.4}$$

Note that $\underline{x}_{P_N}^T \underline{y}_{P_N} = x_{P_N} y_N = \underline{x}_N^T \underline{y}_{P_N}$ and similarly for $\underline{x}_{P_N^{\perp}}^T \underline{y}_{P_N^{\perp}}$. The projection of $\underline{x}_N$ on the subspace $M_{\underline{x}_{N[i,m]}}$ is now decomposed by component projection matrix P_N to obtain

$$P_{\underline{x}_{N[i,m]}} \underline{x}_N = P_{\underline{x}_{N[im]}} \underline{x}_{P_N} + P_{\underline{x}_{N[i,m]}} \underline{x}_{P_N^{\perp}} \tag{6.5}$$

Multiplication of $(I-P_N)$ and the matrix $X_{\underline{x}_{N[i,m]}}$ yields the so-called oblique matrix

$$C_{\underline{x}_{N[i,m]}} = (I - P_N) X_{\underline{x}_{N[i,m]}} \tag{6.6}$$

whose last row is the zero row vector. We define the oblique projection operator to be

$$Q_{\underline{x}_{N[i,m]}} = X_{\underline{x}_{N[i,m]}} \left\{ C_{\underline{x}_{N[i,m]}}^T C_{\underline{x}_{N[i,m]}} \right\}^{-1} C_{\underline{x}_{N[i,m]}}^T \tag{6.7}$$

and its associated orthogonal complement by

$$Q_{\underline{x}_{N[i,m]}}^{\perp} = I - Q_{\underline{x}_{N[i,m]}} \tag{6.8}$$

Upon inspection of expression (6.7), we see that the application of the oblique projection operator to the vector $\underline{x}_N$ implicitly possesses the solution of the prediction coefficients at the N-1st stage.

After simple algebraic manipulation, relationship (6.5) can be expressed as

$$P_{\underline{x}_{N[i,m]}} \underline{x}_N = Q_{\underline{x}_{N[i,m]}} \underline{x}_N + P_{\underline{x}_{N[i,m]}} P_N Q_{\underline{x}_{N[i,m]}}^{\perp} \underline{x}_N \tag{6.9}$$

The orthogonal complement projection of $\underline{x}_N$ can be expressed as

$$P^{\perp}_{\underline{x}_{N[1,m]}} \underline{x}_N = \underline{x}_N - Q_{\underline{x}_{N[i,m]}} \underline{x}_N - P_{\underline{x}_{N[i,m]}} P_N Q^{\perp}_{\underline{x}_{N[i,m]}} \underline{x}_N \qquad (6.10a)$$

which can be further developed to the form

$$P^{\perp}_{\underline{x}_{N[i,m]}} \underline{x}_N = \underline{x}_N - Q_{\underline{x}_{N[i,m]}} \underline{x}_N - P_N Q^{\perp}_{\underline{x}_{N[i,m]}} \underline{x}_N + P^{\perp}_{\underline{x}_{N[i,m]}} P_N Q^{\perp}_{\underline{x}_{N[i,m]}} \underline{x}_N \qquad (6.10b)$$

Considering the relationships (6.3), (6.4) and (6.7), we obtain

$$P^{\perp}_{\underline{x}_{N[i,m]}} \underline{x}_N = \begin{bmatrix} P^{\perp}_{\underline{x}_{N-1[i,m]}} \underline{x}_{N-1} \\ \hline o \ldots\ldots\ldots o \end{bmatrix} + P^{\perp}_{\underline{x}_{N[i,m]}} P_N Q^{\perp}_{\underline{x}_{N[i,m]}} \underline{x}_N \qquad (6.11)$$

Premultiplying $[S^{m+1} \underline{x}_N]^T$ on both sides of expression (6.11) gives the time update recursions of the partial reflection coefficients

$$\Delta_{m+1,N} = \Delta_{m+1,N-1} + [S^{m+1} \underline{x}_N]^T P^{\perp}_{\underline{x}_{N[1,m]}} P_N Q^{\perp}_{\underline{x}_{N[1,m]}} \underline{x}_N \qquad (6.12)$$

where i was taken to be 1. Furthermore, operation of the component projection operator P_N on both sides of expression (6.10a) yields

$$P_N P^{\perp}_{\underline{x}_{N[i,m]}} \underline{x}_N = P_N Q^{\perp}_{\underline{x}_{N[i,m]}} \underline{x}_N - e_N e_N^T P_{\underline{x}_{N[i,m]}} e_N e_N^T Q^{\perp}_{\underline{x}_{N[i,m]}} \underline{x}_N \qquad (6.13a)$$

$$= P_N Q^{\perp}_{\underline{x}_{N[i,m]}} \underline{x}_N [1 - e_N^T P_{\underline{x}_{N[i,m]}} e_N] \qquad (6.13b)$$

Thus we obtain the relationship

$$P_N Q^{\perp}_{\underline{x}_{N[i,m]}} \underline{x}_N = \frac{1}{1 - \gamma_{i,m,N}} P_N P_{\underline{x}_{N[i,m]}} \underline{x}_N \qquad (6.14)$$

where

$$\gamma_{i,m,N} = e_N^T P_{\underline{x}_{N[i,m]}} e_N \qquad (6.15)$$

Directly substituting (6.14) into (6.12), we see that

$$\Lambda_{m+1,N} = \Lambda_{m+1,N-1} + \frac{[S^{m+1}\underline{x}_N]^T P_{\underline{x}_{N[1,m]}^{\perp}} e_N e_N^T P_{\underline{x}_{N[1,m]}^{\perp}} \underline{x}_N}{1 - \gamma_{1,m,N}} \tag{6.16}$$

which simplifies to the form

$$\Lambda_{m+1,N} = \Lambda_{m+1,N-1} + \frac{b_{N-1,m}(N-1)\, c_{N,m}(N)}{1 - \gamma_{1,m,N}} \tag{6.17}$$

Similarly, the time-update for $f^c_{N,m}$ and $f^r_{N,m}$ can be obtained as

$$f^c_{N,m} = f^c_{N-1,m} + \frac{c^2_{N,m}(N)}{1 - \gamma_{1,m,N}} \tag{6.18}$$

$$f^r_{N,m} = f^r_{N-1,m} + \frac{b^2_{N,m}(N)}{1 - \gamma_{0,m-1,N}} \tag{6.19}$$

where

$$\gamma_{0,m-1,N} = e_N^T P_{\underline{x}_{N[0,m-1]}} e_N \tag{6.20}$$

Thus we can use equation (6.17) to update the partial reflection coefficient. Equations (6.18) and (6.19) can be used to update forward and backward prediction errors, respectively.

VII. SUMMARY

A recursive algorithm has been presented for efficiently obtaining an autoregressive (AR) spectral estimate. To achieve a significant computational improvement, prewindowing was applied, and projection operators were utilized in the vector space setting. Normalizations of the order and time update algorithm yield more computational advantage than the unnormalized method. Interested reader may refer to [2], [3].

VIII. REFERENCES

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